CHAPTER 3. BASIC COUNTING

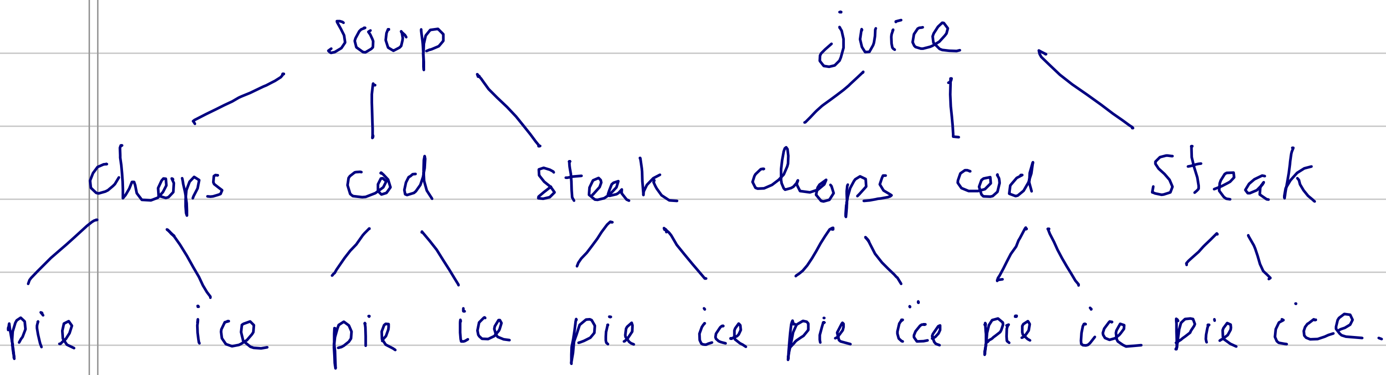
3.1. Permutation.

Example 3.1 A small restaurant has the following menu:

Tomato soup
Fruit Juice
-- x --
Lamb chops
Baked cod
Steak
-- x --
Apple pie
Strawberry ice

How many different three-course meals could you order?

Solution.



There are $2 \times 3 \times 2 = 12$ total number of possible meals.

Example 3.2. In a race with 10 horses, in how many ways can the first three places be filled?

Solution. There are 10 horses that can come first. Whichever horse comes first, there are 9 horses left that can come second. Thus, there are $10 \times 9 = 90$ ways in which the first two places can be filled. In each of these 90 cases, there are 8 horses which can come third. So there are $10 \times 9 \times 8 = 720$ ways in which the first three positions can be filled. $\square$

Multiplication Principle:

If there are K successive choices to be made, and, for $1 \leq i \leq K$, the ith choice can be made in n_i ways, then the total number of ways of making this choices is

$$n_1 \times n_2 \times \cdots \times n_K \quad \left(= \prod_{i=1}^{K} n_i \right).$$

Defn. We call a choice of r objects in order from a set of n objects, a **permutation** of r objects from n.

We denote by $P(n,r)$ the number of permutation of r objects from n.

Theorem:

$$P(n,r) = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

Proof. When we choose r objects in order from n objects, the first object can be chosen in n ways, leaving $n-1$ choices for the second object, and so on. After $r-1$ objects have been chosen there are remain $n-r+1$ objects from which to make the rth choice. By Multiplication Principle, the total number of ways of making these choices is

$$n \times (n-1) \times \cdots \times (n-r-1).$$

$\square$

Defn. The arrangment of different objects using each object exactly once is called a **permutation** of these object.

Corollary. The number of permutations of n objects is $n!$

Notes: We have the following Stirling's formula in approximating $n!$:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Example 3.3. We have five red balls, three yellow balls, and two white balls. In how many different ways can we arrange these balls in a row. (Assume that the balls are identical in size, weight, and texture.)

Solution We have here 10 objects, but they are not all different, so our above Corollary does not apply. The answer is not 10!.

The set of our balls is an example of **"multisets"**. We now label five red balls by 1, 2, 3, 4, 5; 3 yellow balls by 1, 2, 3; and two white balls by 1, 2. Whith this labeling we just distinguish all same-color balls. By corollary we have 10! ways to arrange those lalled balls.

Five red balls could be given five labels in 5! different ways. Three yellow balls can be labelled in 3! ways, and two white balls can be labelled in 2! ways. Thus each arrangement of balls (regardless labels) can be labelled in $5! \cdot 3! \cdot 2!$ ways.
If we have A such arrangments of balls, then

$$A \cdot 5! \cdot 3! \cdot 2! = 10!$$

$$\Rightarrow \quad A = \frac{10!}{5!\,3!\,2!}. \qquad \square$$

Theorem. Let $n, k, a_1, a_2, \dots, a_k$ be nonnegative integers satisfying $a_1 + \dots + a_k = n$. Consider a multiset of n objects, in which a_i objects are of type i, $\forall\, i = 1, \dots, k$. Then the number of ways to linearly order those objects is

$$\frac{n!}{a_1!\, a_2! \cdots a_k!}.$$

$\square$

3.2. Strings over a Finite Alphabet.

We are now studying problems in which we are not simply arranging objects, knowing how many times we can use each objects, but rather construct some "strings", or "words", from a finite set of objects, which we call a finite "alphabet."

We will <u>not</u> require that each symbol occur a specific number of times; we may require that each symbol occur <u>at most once</u>.

Theorem. The number of k-digit strings over an n-element alphabet is n^k.

<u>Proof.</u> We can choose the first "letter" in n different ways. Since a letter may repeat, we can choose the second letter in also n ways, and so on. Thus we have total n^k different ways to choose k letters for our string. $\square$

Example 3.4. How many k-digit positive integers are there?

<u>Hint.</u> $9 \cdot 10^{k-1}$.

We next introduce a powerful method in counting: the "bijective method."

Definition Let X and Y be two finite sets, and let $f: X \to Y$ be a mapping so that

(1) if $f(a) = f(b)$, then $a = b$, and

(2) for all $y \in Y$ there is an $x \in X$ s.t.
$$f(x) = y,$$
then we say that f is a bijection from X onto Y. Equivalently, f is a bijection from X onto Y if for any $y \in Y$, there exists a unique $x \in X$ s.t.
$$f(x) = y.$$

Definition. If f satisfies condition (1) above, then we say that f is one-to-one or injective. If f satisfies condition (2), then f is onto or surjective.

Proposition. Let X and Y are two finite sets. If there exists a bijection from X onto Y, then X and Y have the same number of elements. $\square$

Example 3.5. Prove that the number of all subsets of an n-element set is 2^n.

Proof:

Let $S = \{a_1, a_2, \ldots, a_n\}$ be an n-element set.

We will construct a bijection between subsets of S and binary strings of length n (i.e. the alphabet is $\{0,1\}$).

Let A be a subset of S, $f(A)$ is a string defined as follows. The ith digit of $f(A)$ is 0 if $a_i \notin A$, and is 1 if $a_i \in A$.

It is easy to see that f is a bijection. Thus # of subsets of S equals the number of binary strings of length n, i.e. 2^n. $\square$

3.3 COMBINATIONS.

We are now interested in counting the number of ways of choosing objects from a set when the order of selection does not matter.

Example 3.6. A doubles team is to be selected from a squad of six tennis players. How many different teams can be selected?

Solution.

We have seen that we can choose two players, in order, from a squad of six players in $P(6,2) = 6 \times 5 = 30$ ways. However, this counts each team of two players twice, since for example, choosing first player A and then player B leads to the same team as first choosing

B and then A. Thus the number of different double teams is $30/2 = 15$.
We have to divide by 2 because a particular team of two players can be chosen, in order, in $2! = 2$ diffent ways.

Example 3.7 How many different poker hands of five cards can be drawn from a pack of 52 cards?

Answer: $\frac{P(52,5)}{5!} = \frac{52!}{47!\,5!} = 2{,}598{,}960$.

Defn.
We call a selection of r objects from n objects, when the order does not matter, a **combination** of r objects from n.

We denote by $C(n,r)$ the number of combinatorics of r objects from n.

Theorem. For all $r, n \in \mathbb{N}$, with $r \leq n$,

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!}$$

Notes: We usually use the notation $\binom{n}{r}$ for $\frac{n!}{(n-r)!\,r!}$, and it is read "n choose r".

Basic Properties:

① $\binom{n}{k} = \binom{n}{n-k}$.

② $\binom{n}{0} = \binom{n}{n} = 1$.

Proof. ① LHS = the number of k-subsets of an n-element set S.

RHS = the number of $(n-k)$-subsets of S.

We have a bijection between these two collections of subset by taking the complement.

② $\binom{n}{0}$ is the number of 0-subset of S. However we only have one such subset, the empty set. $\square$

Example 3.8. Consider a sequence $1, 2, \ldots, 100$. How many different ways to select a subsequence of 5 terms, such that it does not contain two consecutive integers?

Solution:

The difficulty is to make sure that we do not choose two consecutive integers. It looks impossible to count these subsequences directly. Assume that we pick a subsequence $1 \leq a_1 < a_2 < a_3 < a_4 < a_5 \leq 100$. The condition "there has two consecutive integers" is equivalent to

$$1 \le a_1 < a_2 - 1 < a_3 - 2 < a_4 - 3 < a_5 - 4 \le 96.$$

Thus we have a bijection between our subsequences and the set of 5-element subsets of $\{1, 2, \dots, 96\}$. Thus the number of 5-subsequences with no two consecutive integers of $\{1, \dots, 100\}$ is $C(96, 5) = \binom{96}{5}$. □

The above trick is also useful when instead of requiring that the chosen elements are far apart, we even allow them to be identical.

Example 3.9. Assume that you are playing a lottery game where five numbers are drawn out of $\{1, 2, \dots, 99\}$, but the numbers drawn are put back into the basket right after being selected. To win a jackpot, one must have played the same multiset of numbers as one drawn (regardless of the order). How many lottery tickets do you have to buy to make sure that you win the jackpot?

Solution. We apply the same trick in the previous example, in reverse. We construct a bijection from set of 5-multisets

$$1 \le a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le 99$$

and the set of 5-elements subsets of $\{1, 2, \dots, 103\}$. Indeed,

$$f(a_1, a_2, a_3, a_4, a_5) = (a_1, a_2 + 1, \dots, a_5 + 4).$$

Thus our answer is $C(103, 5) = \binom{103}{5}$ □

Theorem The number of k-element multisets whose elements all belong to $[n]$ is

$$\binom{n+k-1}{k}$$